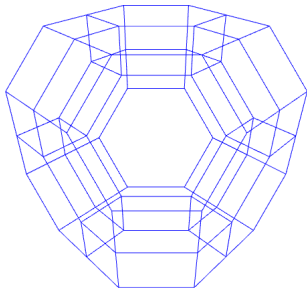


Geometric realizations of the s-permutahedron

Eva Philippe

Sorbonne Université

Journées du GT CombAlg, 3 juillet 2023



Joint work with Rafael S. González
D'León, Alejandro H. Morales, Daniel
Tamayo Jiménez, and Martha Yip

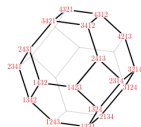
- 1 The s -weak order and the s -permutahedron
- 2 Triangulation of a flow polytope
- 3 Mixed subdivision of a sum of hypercubes
- 4 Polytopal subdivision delimited by an arrangement of tropical hypersurfaces

Motivation

Weak order

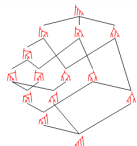


Permutahedron

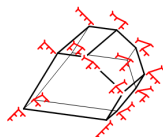


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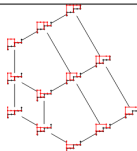
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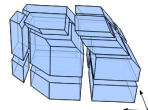
Tamari lattice



Associahedron



ν -Tamari
Préville-Ratelle, Viennot



ν -Associahedron
Ceballos, Padrol, Sarmiento

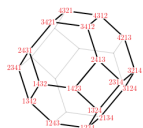
Credit: Pons '19

Motivation

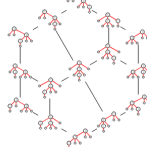
Weak order



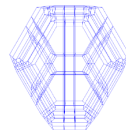
Permutahedron



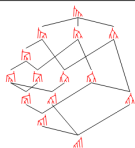
s -Weak order



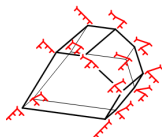
s -Permutahedron



Tamari lattice

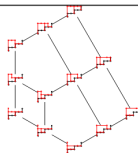


Associahedron



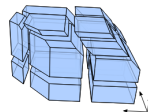
ν -Tamari

Préville-Ratelle, Viennot



ν -Associahedron

Ceballos, Padrol, Sarmiento



Credit: Pons '19

s -decreasing trees (Ceballos-Pons '20)

Let $s = (s_1, \dots, s_n)$ be a (weak) composition (i.e. $s_i \in \mathbb{N}_{\geq 0}$ or in $\mathbb{N}$).

An *s -decreasing tree* is a planar rooted tree on n internal vertices (called nodes), labeled on $[n]$ such that the node labeled i has $s_i + 1$ children and any descendant j of i satisfies $j < i$.

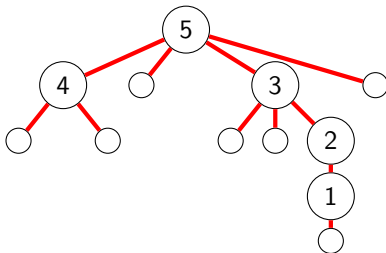


Figure: An $(0, 0, 2, 1, 3)$ -decreasing tree.

s -decreasing trees and Stirling s -permutations (Ceballos-Pons '19)

Let $s = (s_1, \dots, s_n)$ be a composition (i.e. $s_i \in \mathbb{N}_{>0}$).

An s -decreasing tree is associated to a multipermutation of $1^{s_1} \dots n^{s_n}$ that avoids the pattern 121. Such multipermutations are called *Stirling s -permutations*.

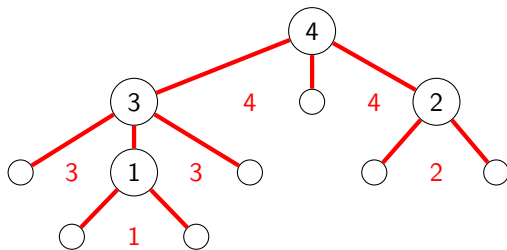
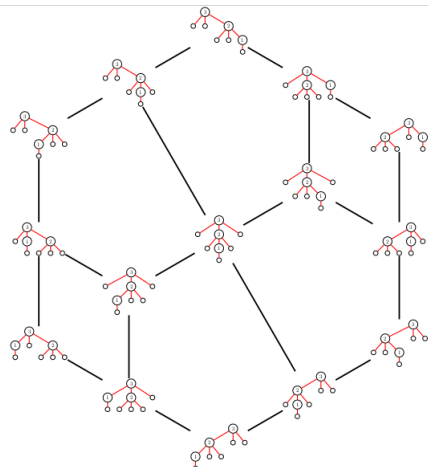


Figure: An $(1, 1, 2, 2)$ -decreasing tree and the corresponding Stirling s -permutation 313442.

The s-weak order



Credit: Ceballos-Pons '19

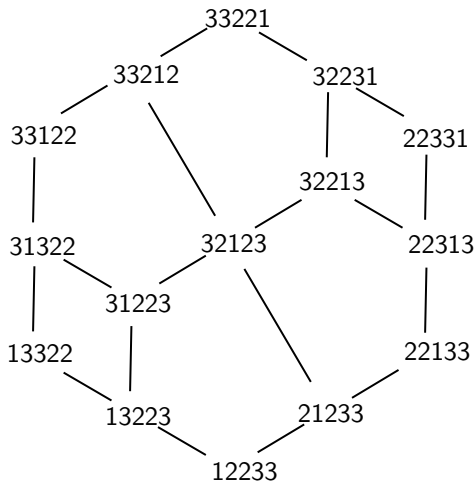


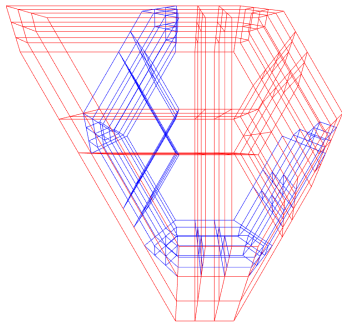
Figure: The $(1, 2, 2)$ -weak order.

Conjecture 1 (Ceballos-Pons '19)

The s -permutahedron can be realized as a polyhedral subdivision of a polytope which is combinatorially isomorphic to a permutahedron.

Conjecture 2 (Ceballos-Pons '19)

If s has no zeros, there exists a geometric realization of the s -permutahedron such that the s -associahedron can be obtained from it by removing certain facets.



Credit: Ceballos-Pons '19

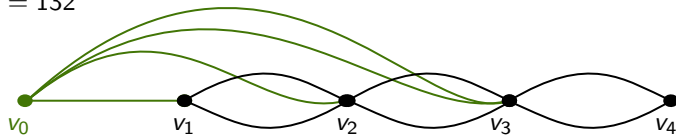
- 1 The s -weak order and the s -permutahedron
- 2 **Triangulation of a flow polytope**
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s-oruga graph

Associated to a composition $s = (s_1, \dots, s_n)$ we consider the graph G_s on vertices $v_0, v_1, \dots, v_{n+1}$ with:

- two edges from v_i to v_{i+1} for $i \in [n]$ and one edge from v_0 to v_1 ,
- $s_{n+1-i} - 1$ edges from v_0 to v_i for $i \in [n]$,
- the *framing* given by ordering incoming and outgoing edges from top to bottom on the drawing.

$s = 132$



A *route* is a path from v_0 to v_{n+1} .

The *flow polytope* $\mathcal{F}_{G_s} = \left\{ (f_e)_{e \in E} \text{ flow of } G \right\} \subset \mathbb{R}^E$ is the convex hull of the indicator vectors of the routes of G_s .

DKK triangulation

We say that two routes P, Q of G are *coherent* with respect to the framing if they "do not cross".

For $C \in \mathcal{C}^{\max}$ (set of maximal cliques of coherent routes), Δ_C denotes the simplex with vertices the indicator vectors of the routes in C .

Theorem (Danilov-Karzanov-Koshevoy, '12)

The simplices $\{\Delta_C \mid C \in \mathcal{C}^{\max}(G, \preceq)\}$ form a (regular) triangulation of $\mathcal{F}_G$, called the *DKK triangulation* of $\mathcal{F}_G$ with respect to the framing $\preceq$.

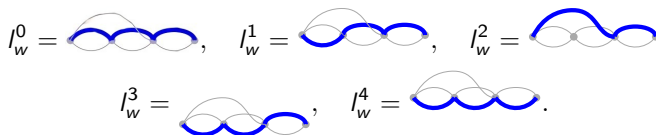


Figure: The maximal clique $\{I_w^0, \dots, I_w^4\}$ corresponding to the $(1, 2, 1)$ -Stirling permutation $w = 3221$.

Theorem (GMPTY, '22)

The s -decreasing trees are in bijection with the simplices of the DKK triangulation of $(\mathcal{F}_{G_s}, \preceq)$.

Moreover, two simplices are adjacent if and only if there is a cover relation in the s -weak order for the corresponding s -decreasing trees.

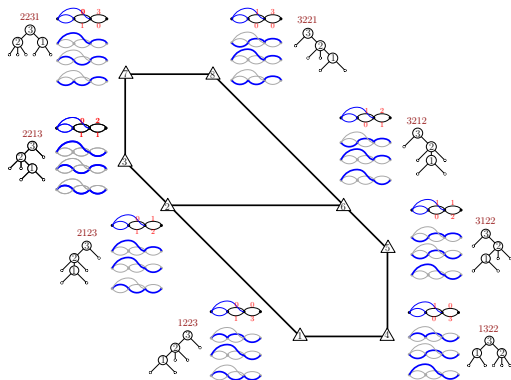
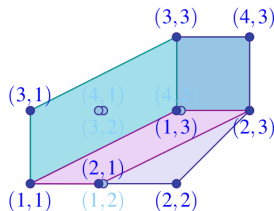


Figure: Dual of the DKK triangulation for $s = (1, 2, 1)$.

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Minkowski sums

- Given polytopes $P_1, \dots, P_k$ in $\mathbb{R}^n$, their *Minkowski sum* is the polytope $P_1 + \dots + P_k := \{\sum x_i \mid x_i \in P_i\}$.
- The *Minkowski cells* of the sum are $\sum B_i$ where B_i is the convex hull of a subset of vertices of P_i .
- A *mixed subdivision* of a Minkowski sum is a collection of Minkowski cells such that their union covers the Minkowski sum and they intersect properly.
- A *fine mixed subdivision* is a minimal mixed subdivision via containment.



Credit: De Loera-Rambau-Santos '19

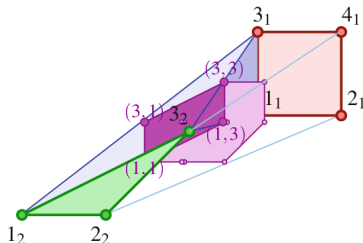
Figure: A (non fine) mixed subdivision of a sum of a square and a triangle.

Cayley Trick

$\mathcal{C}(P_1, \dots, P_k) := \text{conv}(\{e_1\} \times P_1, \dots, \{e_k\} \times P_k) \subset \mathbb{R}^k \times \mathbb{R}^n$ is the *Cayley embedding* of $P_1, \dots, P_k$.

Proposition (The Cayley trick)

The (regular) polytopal subdivisions (resp. triangulations) of $\mathcal{C}(P_1, \dots, P_k)$ are in bijection with the (coherent) mixed subdivisions (resp. fine mixed subdivisions) of $P_1 + \dots + P_k$.



Credit: De Loera-Rambau-Santos '19

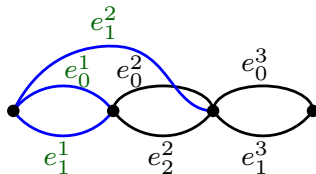
Flow polytopes are Cayley embeddings

Theorem (GMPTY, '22)

The s -decreasing trees are in bijection with the maximal cells of a fine mixed subdivision of the Minkowski sum of hypercubes in $\mathbb{R}^{n-1}$ given by

$$(s_n + 1)\square_{n-1} + \sum_{i=1}^{n-1} (s_i - 1)\square_{i-1}.$$

Proof : The flow polytope of G_s is a Cayley embedding of hypercubes.



Mixed subdivision of hypercubes

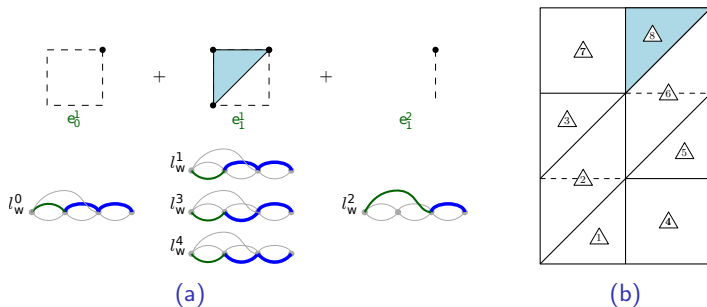
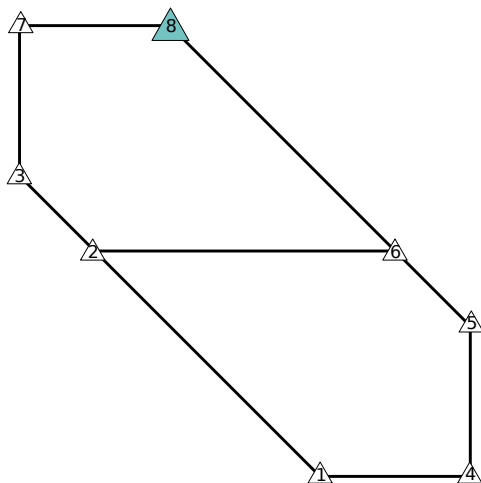
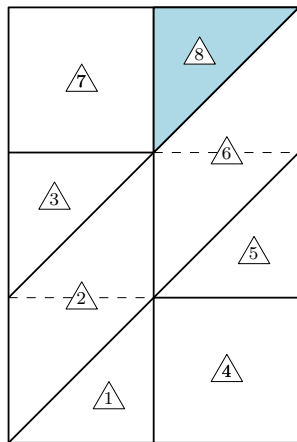


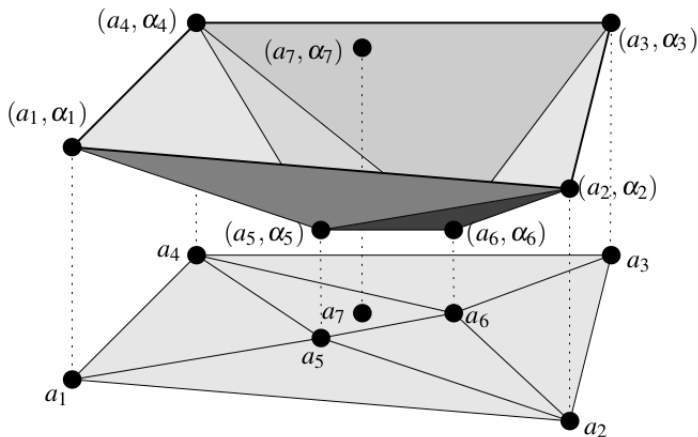
Figure: (a) Summands of the Minkowski cell corresponding to $w = 3221$.
 (b) Mixed subdivision of $2\Box_2 + \Box_1$ realizing the $(1, 2, 1)$ -permutahedron.

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From the mixed subdivision to a dual polyhedral complex



The regular subdivision $\mathcal{S}$ of a point configuration $\mathcal{A} \subset \mathbb{R}^n$ can be obtained as the lower faces of the points of $\mathcal{A}$ lifted by an *admissible height function* α .



Credit: Rambau '96

Danilov-Karzanov-Koshevoy give an explicit admissible height function for DKK triangulations.

Such a lifted configuration is associated to a *tropical polynomial*:

$$F(x) = \bigoplus_{i \in [m]} \alpha^i \odot x^{a^i} = \min \{ \alpha^i + \langle a^i, x \rangle \mid i \in [m] \},$$

that defines the *tropical hypersurface*:

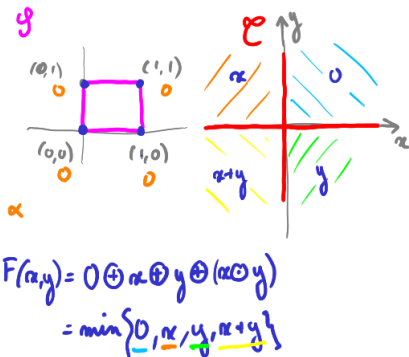
$$\mathcal{T}(F) := \{x \in \mathbb{R}^n \mid \text{the minimum of } F(x) \text{ is attained at least twice}\}.$$

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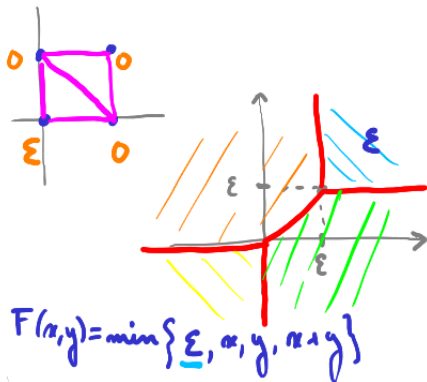


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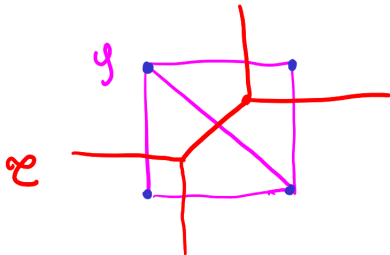
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Theorem (folklore)

There is a bijection between the k -dimensional cells of S and the $(n - k)$ -dimensional cells of $\mathcal{T}(F)$.

The bounded cells of $\mathcal{T}(F)$ corresponds to the interior cells of S .

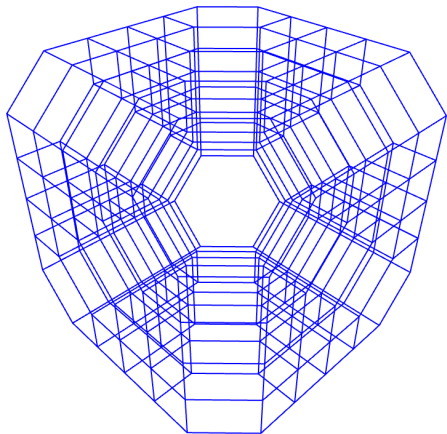


When the point configuration is a Cayley embedding, there is a factorization of the tropical polynomial of the mixed subdivision corresponding to $\mathcal{S}$ via the Cayley trick and we obtain an arrangement of tropical hypersurfaces.

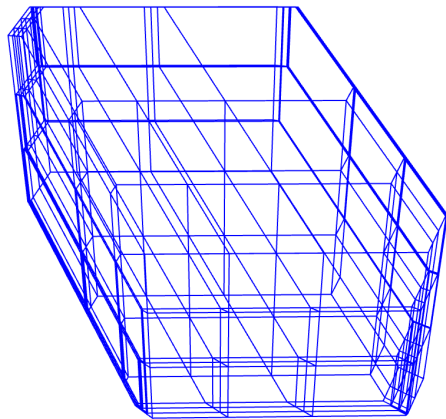
Theorem (GMPTY, '22)

The s -permutahedron can be realized as the bounded cells of an arrangement of tropical hypersurfaces.

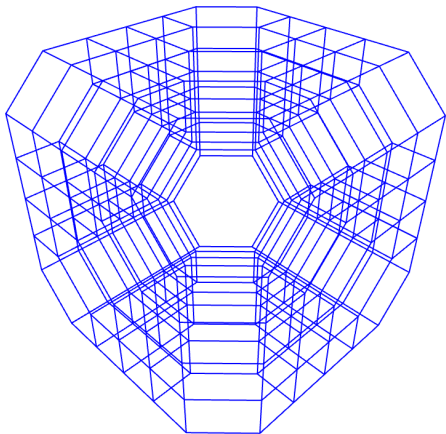
We have explicit coordinates for the vertices and all the faces!



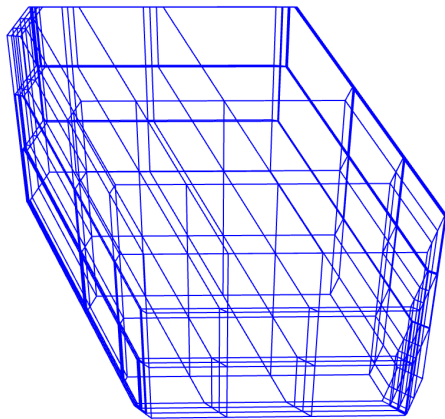
$s = 1114$



$s = 1333$



$s = 1114$



$s = 1333$

Thank you for your attention !

- C. Ceballos and V. Pons. “The s -weak order and s -permutahedra”. *Sém. Lothar. Combin.* 82B (2020), Art. 76, 12.
- V. I. Danilov, A. V. Karzanov, and G. A. Koshevoy. “Coherent fans in the space of flows in framed graphs”. 24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012). *Discrete Math. Theor. Comput. Sci. Proc.* 2012, pp. 481–490.
- J. De Loera, J. Rambau and F. Santos. “Triangulations: structures for algorithms and applications” (Vol. 25). Springer Science & Business Media. (2010) Chapter 9.2.
- M. Joswig, “Essentials of tropical combinatorics”, vol. 219, American Mathematical Society, 2021
- K. Mészáros and A. H. Morales. “Volumes and Ehrhart polynomials of flow polytopes”. *Math. Z.* 293.3–4 (2019), pp. 1369–1401. doi.

